FEA Using the Anti-Symmetry Formulation of ECE EM Theory

Note 2: Reduced Equation System (to Second Degree)

The Equations

In Note 1 of this series, equations were presented for the modeling of a conductive material embedded in a general three-dimensional non-conductive region.

With

$$\rho = 0 \tag{1}$$

the equations were

$$-\frac{\partial}{\partial t}\left(\underline{\nabla}\psi\right) = \omega_0 \underline{\mathbf{A}} \tag{2}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\underline{\mathbf{A}} - \underline{\nabla}\psi\right) + \sigma\mu_0 \frac{\partial}{\partial t}\left(\underline{\mathbf{A}} - \underline{\nabla}\psi\right) = 0 \tag{3}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\underline{\nabla}\phi - \underline{\boldsymbol{\omega}}\phi\right) + \sigma\mu_0 \frac{\partial}{\partial t}\left(\underline{\nabla}\phi - \underline{\boldsymbol{\omega}}\phi\right) = 0 \tag{4}$$

within the conducting region.

In non-conductive regions, the fields were given by

$$-\frac{\partial}{\partial t}\left(\underline{\nabla}\psi\right) = \omega_0 \underline{\mathbf{A}} \tag{2}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\underline{\mathbf{A}} - \underline{\nabla}\psi\right) = 0 \tag{5}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\underline{\nabla}\phi - \underline{\boldsymbol{\omega}}\phi\right) = 0 \tag{6}$$

If we solve equation (2) for $\underline{\mathbf{A}}$ and substitute this value into (3) and (5), we have

$$\underline{\mathbf{A}} = -\frac{1}{\omega_0}\frac{\partial}{\partial t}\left(\underline{\nabla}\psi\right) \tag{7}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\underline{-\frac{1}{\omega_0}\frac{\partial}{\partial t}\left(\nabla\psi\right)} - \underline{\nabla}\psi\right) + \sigma\mu_0 \frac{\partial}{\partial t}\left(\underline{-\frac{1}{\omega_0}\frac{\partial}{\partial t}\left(\nabla\psi\right)} - \underline{\nabla}\psi\right) = 0 \tag{8}$$

$$\left(-\nabla^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(-\frac{1}{\omega_0}\frac{\partial}{\partial t}\left(\underline{\nabla}\psi\right)-\underline{\nabla}\psi\right)=0 \tag{9}$$

Note that we do not have to calculate a wave equation such as (4) and (6), but can use instead, since we have two equivalent definitions for the $\underline{\mathbf{E}}$ field,

$$\underline{\nabla}\phi-\underline{\boldsymbol{\omega}}\phi=\frac{\partial}{\partial t}\left(-\frac{1}{\omega_0}\frac{\partial}{\partial t}\left(\underline{\nabla}\psi\right)-\underline{\nabla}\psi\right) \tag{10}$$

A problem to be addressed is that the equations (8) and (9) are of third order whereas most pde solvers handle only second order equations.

This is easily solved by writing

$$\underline{\mathbf{E}}=\frac{\partial}{\partial t}\left(-\frac{1}{\omega_0}\frac{\partial}{\partial t}\left(\underline{\nabla}\psi\right)-\underline{\nabla}\psi\right) \tag{11}$$

from which equations (8),(9) and (10) become

$$\left(-\nabla^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\underline{\mathbf{E}}+\sigma\mu_0\frac{\partial\underline{\mathbf{E}}}{\partial t}=0 \tag{12}$$

in the conductive regions, and

$$\left(-\nabla^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\underline{\mathbf{E}}=0 \tag{13}$$

outside the conductive regions.

Equation (10) can be rewritten

$$\underline{\nabla}\phi-\underline{\boldsymbol{\omega}}\phi=\underline{\mathbf{E}} \tag{14}$$

Equations (11), (12), (13), and (14) constitute the reduced equation set of nine equations in nine unknowns. This set of equations bears significant resemblance to the equation set that generated the most stable solution in an earlier finite element analysis.[1]

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[1] Two-Dimensional Finite Element Scheme for Conductive Material Using ECE Quasi-static Electromagnetic Theory – Part 1 Distributed Driving Potential; Douglas W. Lindstrom; A.I.A.S. / T.G.A.

Boundary Conditions

The conditions on the interface between a conductive and non-conductive region which were given by applying the divergence theorem to Gauss's and Coulomb's Law now become, upon substituting equation (7)

$$\iint_S \underline{\mathbf{E}} \bullet \underline{\mathbf{n}} ds = 0 \qquad (15)$$

and

$$\iint_S \left(\underline{\nabla} \times -\frac{1}{\omega_0} \frac{\partial}{\partial t} \left(\underline{\nabla} \psi \right) \right) \bullet \underline{\mathbf{n}} ds = \iint_S \left(\underline{\boldsymbol{\omega}} \times -\frac{1}{\omega_0} \frac{\partial}{\partial t} \left(\underline{\nabla} \psi \right) \right) \bullet \underline{\mathbf{n}} ds = 0 \qquad (17)$$

since the Lindstrom Constraint was used in deriving the original set of equations.[2]

[2] The Antisymmetry Law of Cartan Geometry: Applications to Electromagnetism and Gravitation M. W. Evans, D. W. Lindstrom, and H. Eckardt; A.I.A.S / T.G.A.; Paper 134