# Notes 138(10): Antisymmetry of the Connection, Further Details

The fundamental theorem of Riemann geometry is:

$$[D_\mu, D_\nu] V^\rho = \left( \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \right) V^\sigma - \left( \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right) D_\lambda V^\rho . \quad -(1)$$

If

$$\mu = \nu \quad -(2)$$

$$[D_\mu, D_\nu] V^\rho = 0 . \quad -(3)$$

Therefore

$$\Gamma^\lambda_{00} = \Gamma^\lambda_{11} = \Gamma^\lambda_{22} = \Gamma^\lambda_{33} = 0, \quad -(4)$$

$$T^\lambda_{00} = T^\lambda_{11} = T^\lambda_{22} = T^\lambda_{33} = 0 . \quad -(5)$$

Also,

$$\left.\begin{aligned}
\partial_0 \Gamma^\rho_{0\sigma} - \partial_0 \Gamma^\rho_{0\sigma} + \Gamma^\rho_{0\lambda} \Gamma^\lambda_{0\sigma} - \Gamma^\rho_{0\lambda} \Gamma^\lambda_{0\sigma} &= 0 \\
\partial_1 \Gamma^\rho_{1\sigma} - \partial_1 \Gamma^\rho_{1\sigma} + \Gamma^\rho_{1\lambda} \Gamma^\lambda_{1\sigma} - \Gamma^\rho_{1\lambda} \Gamma^\lambda_{1\sigma} &= 0 \\
\partial_2 \Gamma^\rho_{2\sigma} - \partial_2 \Gamma^\rho_{2\sigma} + \Gamma^\rho_{2\lambda} \Gamma^\lambda_{2\sigma} - \Gamma^\rho_{2\lambda} \Gamma^\lambda_{2\sigma} &= 0 \\
\partial_3 \Gamma^\rho_{3\sigma} - \partial_3 \Gamma^\rho_{3\sigma} + \Gamma^\rho_{3\lambda} \Gamma^\lambda_{3\sigma} - \Gamma^\rho_{3\lambda} \Gamma^\lambda_{3\sigma} &= 0
\end{aligned}\right] \quad -(6)$$

The only non-zero connections are:

$$\left.\begin{aligned}
&\Gamma^\lambda_{01} = -\Gamma^\lambda_{10}, \quad \Gamma^\lambda_{02} = -\Gamma^\lambda_{20}, \quad \Gamma^\lambda_{03} = -\Gamma^\lambda_{30} \\
&\Gamma^\lambda_{12} = -\Gamma^\lambda_{21}, \quad \Gamma^\lambda_{13} = -\Gamma^\lambda_{31} \\
&\Gamma^\lambda_{23} = -\Gamma^\lambda_{32} .
\end{aligned}\right] \quad -(7)$$

Therefore in the Riemann tensor:

$$R^\rho{}_{\sigma\mu\nu} := \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \quad -(8)$$

$$\mu \neq \nu \quad -(9)$$

and

$$R^\rho{}_{\sigma\mu\nu} = - R^\rho{}_{\sigma\nu\mu} . \quad -(10)$$

2. The other symmetries are:

$$T^\lambda_{\mu\nu} = -T^\lambda_{\nu\mu}, \quad \text{— (11)}$$

$$\Gamma^\lambda_{\mu\nu} = -\Gamma^\lambda_{\nu\mu}, \quad \text{— (12)}$$

$$\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} = -\left(\partial_\nu \Gamma^\rho_{\mu\sigma} - \partial_\mu \Gamma^\rho_{\nu\sigma}\right), \quad \text{— (13)}$$

$$\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} = -\left(\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}\right), \quad \text{— (14)}$$

$$\Gamma^\rho_{\nu\sigma} = -\Gamma^\rho_{\sigma\nu} \quad \text{— (15)}$$

$$\Gamma^\rho_{\mu\sigma} = -\Gamma^\rho_{\sigma\mu} \quad \text{— (16)}$$

$$\Gamma^\rho_{\mu\lambda} = -\Gamma^\rho_{\lambda\mu} \quad \text{— (17)}$$

$$\Gamma^\lambda_{\nu\sigma} = -\Gamma^\lambda_{\sigma\nu} \quad \text{— (18)}$$

$$\Gamma^\rho_{\nu\lambda} = -\Gamma^\rho_{\lambda\nu} \quad \text{— (19)}$$

$$\Gamma^\lambda_{\mu\sigma} = -\Gamma^\lambda_{\sigma\mu} \quad \text{— (20)}$$

## The Error in the Twentieth Century Cosmology

This was to assume that the connection could be symmetric and non-zero. This is a glaring error because it assumes that there is a non-zero symmetric commutator. This assumption was used to write the incorrect equation:

$$[D_\mu, D_\nu] V^\rho = ?\left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right) V^\sigma \neq 0 \quad \text{— (21)}$$

In this equation there is no indication of the symmetry of the connection, whereas the correct eq. (1) fixes the antisymmetry (12) through:

$$[D_\mu, D_\nu] V^\rho = -\Gamma^\lambda_{\mu\nu} + \cdots \qquad -(22)$$

The commutator $[D_\mu, D_\nu]$ and the connection $\Gamma^\lambda_{\mu\nu}$ must both be antisymmetric.

In the incorrect eqn. (21), there is nothing to indicate this, and the error was compounded by assuming that:

$$\Gamma^\lambda_{\mu\nu} \overset{?}{=} \frac{1}{2}\left(\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\nu}\right) + \frac{1}{2}\left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\nu}\right) \qquad -(23)$$

in which:

$$\Gamma^\lambda_{\mu\nu}(S) \overset{?}{=} \Gamma^\lambda_{\nu\mu}(S) \qquad -(24)$$

and

$$\Gamma^\lambda_{\mu\nu}(A) = -\Gamma^\lambda_{\nu\mu}(A). \qquad -(25)$$

The correct eq. (22) shows that eq. (25) is the correct antisymmetry.

Scientific History

The basic error is so glaring:

$$\boxed{[D_\mu, D_\nu] V^\rho \overset{?}{=} [D_\nu, D_\mu] V^\rho \overset{?}{\neq} 0} \qquad -(26)$$

that some research is needed into why it was made, and why it was repeated for ninety years.