140(14): Viscosity Effects in Fluid Flow

For an inviscid fluid, as in previous notes:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) = -\underline{\nabla}p - \rho\underline{\nabla}\phi \quad -(1)$$

The viscous force, $\underline{f}_v$, is added to the right hand side of eq. (1) to produce:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) = -\underline{\nabla}p - \rho\underline{\nabla}\phi + \underline{f}_v \quad -(2)$$

The most general form of second derivatives that can occur in a vector equation is a linear combination of terms $\nabla^2\underline{v}$ and $\underline{\nabla}(\underline{\nabla}\cdot\underline{v})$. Therefore:

$$\underline{f}_v = \mu\nabla^2\underline{v} + (\mu+\mu')\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) \quad -(3)$$

where μ and μ' are coefficients. From eqs. (2) and (3):

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) = -\underline{\nabla}p - \rho\underline{\nabla}\phi + \mu\nabla^2\underline{v} + (\mu+\mu')\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) \quad -(4)$$

The vorticity is defined as:

$$\underline{\Omega} = \underline{\nabla}\times\underline{v} \quad -(5)$$

Using the identity:

$$(\underline{v}\cdot\underline{\nabla})\underline{v} = (\underline{\nabla}\times\underline{v})\times\underline{v} + \frac{1}{2}\nabla(\underline{v}\cdot\underline{v}) \quad -(6)$$

and in an incompressible fluid:

$$\underline{\nabla}\cdot\underline{v} = 0 \quad -(7)$$

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$$\frac{\partial \underline{\Omega}}{\partial t} + \underline{\nabla} \times (\underline{\Omega} \times \underline{v}) = \frac{\mu}{\rho} \nabla^2 \underline{\Omega} \quad -(8)$$

which is the equation of motion of a viscous fluid.

The inviscid fluid is

$$\frac{\partial \underline{\Omega}}{\partial t} + \underline{\nabla} \times (\underline{\Omega} \times \underline{v}) = \underline{0} \quad -(9)$$

As in previous notes these bear a similarity to the equations of electrodynamics.